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ON A PALEY-WIENER THEOREM AND A q-SAMPLING FORMULA FOR THE q-DUNKL TRANSFORM

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Abstract

In this paper, we present some new elements of q-harmonic analysis associated with a q-Dunkl operator. Next, we prove a Paley-Wiener theorem for a q-Dunkl transform and characterize a q-Dunkl Paley-Wiener space. Finally, we prove a q-Dunkl sampling theorem, using $\pm q^n$, $n \in \mathbb{Z}$, as sampling points.

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1. Introduction

The celebrated classical Paley-Wiener theorem asserts that for a positive real a, the Paley-Wiener space

$$PW_a = \left\{ f : \quad f(x) = \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} e^{ixt} u(t) dt, \qquad u \in \mathcal{D}(-a,a) \right\}$$

is composed by functions analytically extendable into complex plane as entire functions of exponential type at most a. Here, $\mathcal{D}(-a,a)$ is the space of C^∞ functions on $\mathbb{R}$ with compact supports in $(-a,a)$. Each element of PW_a is called band-limited signal.

The famous Whittaker-Shannon-Kotel'nikov sampling theorem asserts that every band-limited signal can be written as

$$f(x) = \sum_{n=-\infty}^{\infty} f\left(\frac{\pi}{a} n\right) \frac{\sin(ax - \pi n)}{ax - \pi n}.$$

In the literature, these theorems are known to hold for more general transforms in classical analysis as well as in Quantum Calculus (see [1], [2], [4], [5], [7], [8], [10], [12], [15], [16] and references therein).

Recently, a q-analogue of the Dunkl transform on the real line (see [3]) has been introduced and some elements of q-Dunkl harmonic analysis have been discussed.

In this paper, we introduce and study the generalized q-Dunkl translation operator, we give two characterizations of the q-Dunkl Paley-Wiener space and prove a q-sampling theorem, using $\pm q^n, \quad n \in \mathbb{Z}$ as sampling points.

The paper is organized as follows: in Section 2, we present some preliminary notions and notations useful in the sequel. In Section 3, we recall some results and properties from the theory of the q-Dunkl operator and the q-Dunkl transform (see [3]). Next, in Section 4, we define the generalized q-Dunkl translation operator and discuss its properties. Section 5 is devoted to prove a Paley-Wiener theorem, to give two characterizations of the q-Dunkl Paley-Wiener space and to prove a q-Dunkl sampling theorem.

2. Notations and preliminaries

Throughout this paper, we assume $q \in (0,1)$, and we refer to the general reference [9] for the definitions, notations and properties of the q-shifted factorials and the q-hypergeometric functions.

We write $\mathbb{R}_q = \{\pm q^n \ : \ n \in \mathbb{Z}\}$, $\widetilde{\mathbb{R}}_q = \mathbb{R}_q \cup \{0\}$, $\mathbb{R}_{q,+} = \{q^n \ : \ n \in \mathbb{Z}\}$,

$$[x]_q = \frac{1-q^x}{1-q}, \qquad x \in \mathbb{C} \quad \text{and} \quad [n]_q! = \frac{(q;q)_n}{(1-q)^n}, \qquad n \in \mathbb{N}.$$

The q^2-analogue differential operator is (see [13, 14])

$$\partial_q(f)(z) = \begin{cases} \dfrac{f\left(q^{-1}z\right) + f\left(-q^{-1}z\right) - f\left(qz\right) + f\left(-qz\right) - 2f(-z)}{2(1-q)z} & \text{if } z \neq 0, \\ \lim\limits_{x \to 0} \partial_q(f)(x) \qquad (\text{in } \mathbb{R}_q) & \text{if } z = 0. \end{cases}$$

Remark that if f is differentiable at z, then $\lim\limits_{q \to 1} \partial_q(f)(z) = f'(z)$.

The q-Jackson integrals are defined by (see [11])

$$\int_0^a f(x)d_qx = (1-q)a\sum_{n=0}^{\infty} q^n f(aq^n), \quad \int_a^b f(x)d_qx = \int_0^b f(x)d_qx - \int_0^a f(x)d_qx,$$

$$\int_0^{\infty} f(x)d_qx = (1-q)\sum_{n=-\infty}^{\infty} q^n f(q^n), \quad \int_{-\infty}^{\infty} f(x)d_qx = \int_0^{\infty} f(x)d_qx + \int_0^{\infty} f(-x)d_qx,$$

provided the sums converge absolutely.

In what follows, we use the notations of the following sets and spaces:

• $\mathcal{S}_q(\mathbb{R}_q)$ is the space of functions f defined on $\mathbb{R}_q$ satisfying

$$\forall n,m \in \mathbb{N}, \qquad P_{n,m,q}(f) = \sup_{x\in\mathbb{R}_q} \mid x^m \partial_q^n f(x) \mid < +\infty$$

and $\lim\limits_{x\to 0} \partial_q^n f(x)$(in $\mathbb{R}_q$) exists.

• $\mathcal{D}_q(\mathbb{R}_q)$ is the subspace of $\mathcal{S}_q(\mathbb{R}_q)$ composed of functions with compact support in $\mathbb{R}_q$ and for $A \subset \mathbb{R}$, $\mathcal{D}_q(A)$ is the subspace of $\mathcal{D}_q(\mathbb{R}_q)$ constituted of functions with supports in A.

• $L_q^{\infty}(\mathbb{R}_q) = \left\{ f : \|f\|_{\infty,q} = \sup\limits_{x\in\mathbb{R}_q} |f(x)| < \infty \right\}$.

• $L_{\alpha,q}^{p}(\mathbb{R}_q) = \left\{ f : \|f\|_{p,\alpha,q} = \left(\int_{-\infty}^{\infty} |f(x)|^p |x|^{2\alpha+1} d_qx \right)^{\frac{1}{p}} < \infty \right\}$.

3. The q-Dunkl operator and the q-Dunkl transform

In this section, we collect some basic properties of the q-Dunkl operator and the q-Dunkl transform introduced in [3], useful in the sequel.

For $\alpha \geq -\frac{1}{2}$, the q-Dunkl operator is defined by

$$\Lambda_{\alpha,q}(f)(x) = \partial_q \left[H_{\alpha,q}(f) \right](x) + [2\alpha+1]_q \frac{f(x)-f(-x)}{2x},$$

where

$$H_{\alpha,q} : f = f_e + f_o \longmapsto f_e + q^{2\alpha+1} f_o.$$

It leaves the spaces $\mathcal{S}_q(\mathbb{R}_q)$ and $\mathcal{D}_q(\mathbb{R}_q)$ invariant and satisfies

$$\int_{-\infty}^{+\infty} \Lambda_{\alpha,q}(f)(x) g(x) |x|^{2\alpha+1} d_qx = -\int_{-\infty}^{+\infty} \Lambda_{\alpha,q}(g)(x) f(x) |x|^{2\alpha+1} d_qx. \quad (1)$$

Furthermore, we have the following general result.

Lemma 1. *For $a>0$, if $\int_{-a}^{a}\Lambda_{\alpha,q}(f)(x)g(x)|x|^{2\alpha+1}d_qx$ exists, then*

$$\int_{-a}^{a}\Lambda_{\alpha,q}(f)(x)g(x)|x|^{2\alpha+1}d_qx = 2a^{2\alpha+1}\left[f_e(q^{-1}a)g_o(a)+f_o(a)g_e(q^{-1}a)\right]$$
$$-\int_{-a}^{a}\Lambda_{\alpha,q}(g)(x)f(x)|x|^{2\alpha+1}d_qx. \tag{2}$$

Proof. The result follows from [3, Lemma 2], the definition of the q-Jackson integral and the properties of the q-Dunkl operator $\Lambda_{\alpha,q}$. ∎

It was shown in [3] that for each $\lambda\in\mathbb{C}$, the function

$$\psi_{\alpha,q}(x,\lambda)=j_\alpha(\lambda x;q^2)+\frac{i\lambda x}{[2\alpha+2]_q}j_{\alpha+1}(\lambda x;q^2) \tag{3}$$

is the unique solution of the q-differential-difference equation:

$$\begin{cases}\Lambda_{\alpha,q}(f)=i\lambda f\\ f(0)=1,\end{cases}$$

where $j_\alpha(.;q^2)$ is the normalized third Jackson's q-Bessel function given by

$$j_\alpha(x;q^2)=\sum_{n=0}^{\infty}(-1)^n\frac{q^{n(n+1)}}{(q^2;q^2)_n(q^{2(\alpha+1)};q^2)_n}((1-q)x)^{2n}. \tag{4}$$

The function $\psi_{\alpha,q}$ has a unique extension to $\mathbb{C}\times\mathbb{C}$ and satisfies the following properties:

1) For $n\in\mathbb{N},\ x,\lambda\in\mathbb{C}$,

$$\Lambda^n_{\alpha,q}[\psi_{\alpha,q}(.,\lambda)](x)=(i\lambda)^n\psi_{\alpha,q}(x,\lambda). \tag{5}$$

2) $\psi_{\alpha,q}(x,a\lambda)=\psi_{\alpha,q}(ax,\lambda)=\psi_{\alpha,q}(\lambda,ax)$, for all $a,x,\lambda\in\mathbb{C}$.

3) For all $n\in\mathbb{N}$, we have

$$\mid\partial_q^n[\psi_{\alpha,q}(.,\lambda)](x)\mid\leq\frac{4\mid\lambda\mid^n}{(q;q)_\infty},\quad\text{for all }\lambda,x\in\mathbb{R}_q. \tag{6}$$

In particular,

$$\mid\psi_{\alpha,q}(x,\lambda)\mid\leq\frac{4}{(q;q)_\infty},\quad\text{for all }x,\lambda\in\mathbb{R}_q. \tag{7}$$

4) From Lemma 1, we have for all $x,y\in\mathbb{C}$ such that $y\neq -x$,

$$\int_{-a}^{a}\psi_{\alpha,q}(t,x)\,\psi_{\alpha,q}(t,y)|t|^{2\alpha+1}d_qt=\frac{2a^{2\alpha+2}}{[2\alpha+2]_q(y+x)}$$
$$\times\left\{yj_{\alpha+1}(ay;q^2)j_\alpha\left(\frac{ax}{q};q^2\right)+xj_{\alpha+1}(ax;q^2)j_\alpha\left(\frac{ay}{q};q^2\right)\right\}. \tag{8}$$

The q-Dunkl transform $F_D^{\alpha,q}$ is defined on $L^1_{\alpha,q}(\mathbb{R}_q)$ (see [3]) by

$$F_D^{\alpha,q}(f)(\lambda) = \frac{c_{\alpha,q}}{2}\int_{-\infty}^{+\infty} f(x)\psi_{\alpha,q}(x,-\lambda)|x|^{2\alpha+1}d_qx, \text{ with } c_{\alpha,q} = \frac{(1+q)^{-\alpha}}{\Gamma_{q^2}(\alpha+1)}.$$

It satisfies the following properties:

- For all $f \in L^1_{\alpha,q}(\mathbb{R}_q)$, we have $\|F_D^{\alpha,q}(f)\|_{\infty,q} \leq \dfrac{2c_{\alpha,q}}{(q;q)_\infty}\|f\|_{1,\alpha,q}$.
- For all $f \in L^1_{\alpha,q}(\mathbb{R}_q)$, such that $xf \in L^1_{\alpha,q}(\mathbb{R}_q)$,

$$F_D^{\alpha,q}(\Lambda_{\alpha,q}f)(\lambda) = i\lambda F_D^{\alpha,q}(f)(\lambda) \text{ and } \Lambda_{\alpha,q}(F_D^{\alpha,q}(f)) = -iF_D^{\alpha,q}(xf). \tag{9}$$

- Plancherel theorem: $F_D^{\alpha,q}$ is an isomorphism from $L^2_{\alpha,q}(\mathbb{R}_q)$ (resp. $\mathcal{S}_q(\mathbb{R}_q)$) onto itself and for $f \in L^2_{\alpha,q}(\mathbb{R}_q)$, we have

$$\|F_D^{\alpha,q}(f)\|_{2,\alpha,q} = \|f\|_{2,\alpha,q} \tag{10}$$

and

$$(F_D^{\alpha,q})^{-1}(f)(x) = \frac{c_{\alpha,q}}{2}\int_{-\infty}^{+\infty} f(\lambda)\psi_{\alpha,q}(x,\lambda)|\lambda|^{2\alpha+1}d_q\lambda. \tag{11}$$

4. q-Translation operator associated with the q-Dunkl operator

Definition 1. The generalized q-Dunkl translation operator is defined for $f \in L^2_{\alpha,q}(\mathbb{R}_q)$ and $x, y \in \mathbb{R}_q$ by

$$T_y^{\alpha,q}(f)(x) = \frac{c_{\alpha,q}}{2}\int_{-\infty}^{\infty} F_D^{\alpha,q}(f)(\lambda)\psi_{\alpha,q}(x,\lambda)\psi_{\alpha,q}(y,\lambda)|\lambda|^{2\alpha+1}d_q\lambda, \tag{12}$$

and $T_0^{\alpha,q}(f)(x) = f(x)$.

The following result gives a Taylor formula for the generalized q-Dunkl translation operator.

Proposition 1. *Let $f \in \mathcal{S}_q(\mathbb{R}_q)$ satisfy the following: there exist positive constants C and R such that*

$$\|\Lambda^n_{\alpha,q}f\|_{1,\alpha,q} \leq CR^n, \text{ for all } n \in \mathbb{N}.$$

Then for all $x, y \in \mathbb{R}_q$,

$$T_y^{\alpha,q}(f)(x) = \sum_{n=0}^{+\infty} a_n(\alpha,q^2)\Lambda^n_{\alpha,q}(f)(y)x^n = \sum_{n=0}^{+\infty} a_n(\alpha,q^2)\Lambda^n_{\alpha,q}(f)(x)y^n, \tag{13}$$

where

$$\begin{cases} a_{2n}(\alpha,q^2) = \dfrac{q^{n(n+1)}}{(q^2;q^2)_n(q^{2(\alpha+1)};q^2)_n}(1-q)^{2n} \\ a_{2n+1}(\alpha,q^2) = \dfrac{1}{1-q^{2(\alpha+1)}} \times \dfrac{q^{n(n+1)}}{(q^2;q^2)_n(q^{2(\alpha+2)};q^2)_n}(1-q)^{2n+1}. \end{cases} \tag{14}$$

Proof. Let $f \in \mathcal{S}_q(\mathbb{R}_q)$, satisfying the condition of the proposition and fix $x, y \in \mathbb{R}_q$. On the one hand, from (3) and (4), we have

$$\psi_{\alpha,q}(x,\lambda) = \sum_{n=0}^{+\infty} a_n(\alpha, q^2)(i\lambda x)^n$$

and from the Plancherel theorem, we have

$$\sum_{n=0}^{+\infty} a_n(\alpha, q^2)\Lambda_{\alpha,q}^n(f)(y)x^n = \sum_{n=0}^{+\infty} a_n(\alpha, q^2)x^n F_D^{\alpha,q}\left(F_D^{\alpha,q}(\Lambda_{\alpha,q}^n f)\right)(-y).$$

On the other hand, using the fact that for all $n \in \mathbb{N}$,

$$\|F_D^{\alpha,q}(\Lambda_{\alpha,q}^n f)\|_{\infty,q} \leq C_1 \|\Lambda_{\alpha,q}^n f\|_{1,\alpha,q} \leq C_2 R^n,$$

we deduce that

$$\sum_{n\geq 0} \int_{-\infty}^{\infty} \left|a_n(\alpha, q^2) F_D^{\alpha,q}(\Lambda_{\alpha,q}^n f)(\lambda)\psi_{\alpha,q}(y,\lambda)\lambda^{2\alpha+1}x^n\right| d_q\lambda$$

converges. Then the Fubini theorem achieves the proof. ∎

As an immediate consequence of the previous proposition, we have the following result.

Corollary 1. *If f satisfies the condition of Proposition 1, then for all $x \in \mathbb{R}_q$, the function $T_z^{\alpha,q}(f)(x)$ is entire on $\mathbb{C}$ and for all $z \in \mathbb{C}$,*

$$T_z^{\alpha,q}(f)(x) = \sum_{n=0}^{+\infty} a_n(\alpha, q^2)\Lambda_{\alpha,q}^n(f)(x)z^n.$$

Proposition 2.
1) For $f \in L^1_{\alpha,q}(\mathbb{R}_q) \cup L^2_{\alpha,q}(\mathbb{R}_q)$, we have

$$T_y^{\alpha,q}(f)(x) = T_x^{\alpha,q}(f)(y), x, y \in \widetilde{\mathbb{R}}_q.$$

2) For all $y \in \widetilde{\mathbb{R}}_q$ and all $f \in L^2_{\alpha,q}(\mathbb{R}_q)$ (resp. $\mathcal{S}_q(\mathbb{R}_q)$), we have $T_y^{\alpha,q} f \in L^2_{\alpha,q}(\mathbb{R}_q)$ (resp. $\mathcal{S}_q(\mathbb{R}_q)$) and $\| T_y^{\alpha,q} f \|_{2,\alpha,q} \leq \dfrac{4}{(q;q)_\infty} \| f \|_{2,\alpha,q}$.

3) For $f \in L^2_{\alpha,q}(\mathbb{R}_q)$ and $y \in \widetilde{\mathbb{R}}_q$, we have

$$F_D^{\alpha,q}(T_y^{\alpha,q} f)(\lambda) = \psi_{\alpha,q}(y,\lambda)F_D^{\alpha,q}(f)(\lambda). \tag{15}$$

4) We have the following product formula

$$T_y^{\alpha,q}\left[\psi_{\alpha,q}(.,\lambda)\right](x) = \psi_{\alpha,q}(x,\lambda)\psi_{\alpha,q}(y,\lambda). \tag{16}$$

5) For $f \in \mathcal{S}_q(\mathbb{R}_q), y \in \mathbb{R}_q$, we have $\Lambda_{\alpha,q} T_y^{\alpha,q} f = T_y^{\alpha,q} \Lambda_{\alpha,q} f$.

6) For all $f, g \in \mathcal{S}_q(\mathbb{R}_q)$,

$$\int_{-\infty}^{\infty} T_x^{\alpha,q} f(-y) g(y) |y|^{2\alpha+1} d_q y = \int_{-\infty}^{\infty} f(y) T_x^{\alpha,q} g(-y) |y|^{2\alpha+1} d_q y. \qquad (17)$$

Proof. 1) It is an immediate consequence of the definition of the generalized q-Dunkl translation operator.

2) Let $y \in \widetilde{\mathbb{R}}_q$ and $f \in L^2_{\alpha,q}(\mathbb{R}_q)$(resp. $\mathcal{S}_q(\mathbb{R}_q)$). Since $\psi_{\alpha,q}(.,y)$ is bounded on $\mathbb{R}_q$ (resp. belongs to $\mathcal{S}_q(\mathbb{R}_q)$), we have, by using the Plancherel theorem, $\psi_{\alpha,q}(.,y) F_D^{\alpha,q}(f) \in L^2_{\alpha,q}(\mathbb{R}_q)$ (resp. $\mathcal{S}_q(\mathbb{R}_q)$), and

$$T_y^{\alpha,q}(f) = (F_D^{\alpha,q})^{-1} \left[\psi_{\alpha,q}(.,y) F_D^{\alpha,q}(f)\right] \in L^2_{\alpha,q}(\mathbb{R}_q) \text{ (resp.} \mathcal{S}_q(\mathbb{R}_q)).$$

Furthermore,

$$\|T_y^{\alpha,q}(f)\|_{2,\alpha,q} = \|(F_D^{\alpha,q})^{-1} \left[\psi_{\alpha,q}(.,y) F_D^{\alpha,q}(f)\right] \|_{2,\alpha,q} = \|\psi_{\alpha,q}(.,y) F_D^{\alpha,q}(f)\|_{2,\alpha,q}$$
$$\leq \|\psi_{\alpha,q}(.,y)\|_{\infty,q} \|F_D^{\alpha,q}(f)\|_{2,\alpha,q} \leq \frac{4}{(q;q)_\infty} \|f\|_{2,\alpha,q}.$$

3) The result follows from the previous Plancherel theorem.

4) Proposition 1 and the relations (5) and (7) give the relation.

5) Let $f \in \mathcal{S}_q(\mathbb{R}_q)$. From the properties of the q-Dunkl operator and the generalized q-Dunkl translation operator, we have $\Lambda_{\alpha,q}(f)$, $T_y^{\alpha,q}(f)$, $\Lambda_{\alpha,q} T_y^{\alpha,q}(f)$ and $T_y^{\alpha,q} \Lambda_{\alpha,q}(f)$ are in $\mathcal{S}_q(\mathbb{R}_q)$. So, by the help of the relations (9) and (15), we get

$$F_D^{\alpha,q}\left[\Lambda_{\alpha,q} T_y^{\alpha,q}(f)\right](\lambda) = i\lambda F_D^{\alpha,q}[T_y^{\alpha,q}(f)](\lambda) = i\lambda \psi_{\alpha,q}(y,\lambda) F_D^{\alpha,q}(f)(\lambda)$$

and

$$F_D^{\alpha,q}\left[T_y^{\alpha,q} \Lambda_{\alpha,q}(f)\right](\lambda) = \psi_{\alpha,q}(y,\lambda) F_D^{\alpha,q}[\Lambda_{\alpha,q}(f)](\lambda) = i\lambda \psi_{\alpha,q}(y,\lambda) F_D^{\alpha,q}(f)(\lambda).$$

The Plancherel theorem achieves the proof of the relation.

6) Let $f, g \in \mathcal{S}_q(\mathbb{R}_q)$. It is easy to see that for all $t, x, y \in \mathbb{R}_q$, we have

$$|F_D^{\alpha,q}(f)(t) \psi_{\alpha,q}(t,x) \psi_{\alpha,q}(t,-y) g(y)| \leq \frac{4}{(q;q)_\infty} \|F_D^{\alpha,q}(f)\|_{\infty,q} |\psi_{\alpha,q}(t,x)||g(y)|.$$

Hence, since $\psi_{\alpha,q}(.,x)$ and g are in $\mathcal{S}_q(\mathbb{R}_q) \subset L^1_{\alpha,q}(\mathbb{R}_q)$, the Fubini theorem gives the desired relation. ∎

5. q-Dunkl Paley-Wiener theorem

In this section, for $a \in \mathbb{R}_{q,+}$ and $\alpha \geq -\frac{1}{2}$, we introduce the q-Dunkl Paley-Wiener space $PW_{q,a}^{\alpha}$ as

$$PW_{q,a}^{\alpha} = \left\{ f(x) = \frac{c_{\alpha,q}}{2} \int_{-a}^{a} u(t)\psi_{\alpha,q}(t,x)|t|^{2\alpha+1} d_q t, \quad u \in \mathcal{D}_q([-a,a]) \right\}.$$

PROPOSITION 3.

1) The q-Dunkl transform $F_D^{\alpha,q}$ is an isomorphism from $PW_{q,a}^{\alpha}$ onto $\mathcal{D}_q([-a,a])$.

2) Every element of the q-Dunkl Paley-Wiener space $PW_{q,a}^{\alpha}$ is the restriction on $\mathbb{R}_q$ of an entire function on $\mathbb{C}$ of exponential type.

Proof. 1) It follows from the definition of $PW_{q,a}^{\alpha}$ and the Plancherel theorem.

2) Let $f \in PW_{q,a}^{\alpha}$. There exists $u \in \mathcal{D}_q([-a,a])$ such that for all $x \in \mathbb{R}_q$,

$$f(x) = \frac{c_{\alpha,q}}{2} \int_{-a}^{a} u(t)\psi_{\alpha,q}(t,x)|t|^{2\alpha+1} d_q t. \tag{18}$$

Since for all $t \in \mathbb{R}_q \cap [-a,a]$, the function $\psi_{\alpha,q}(t,.)$ is entire on $\mathbb{C}$ and satisfies for all $R > 0$ and all $z \in \mathbb{C}$ such that $|z| < R$,

$$\left| u(t)\psi_{\alpha,q}(t,z)|t|^{2\alpha+1} \right| \leq a^{2\alpha+1} \|u\|_{\infty,q} \psi_{\alpha,q}(R,-ia),$$

then the function $\int_{-a}^{a} u(t)\psi_{\alpha,q}(t,z)|t|^{2\alpha+1} d_q t$ is entire on $\mathbb{C}$ and f is extendable to an entire function on $\mathbb{C}$.

On the other hand, making a proof as in [6, Proposition 2], one can show that for all $z \in \mathbb{C}$, and all $t \in \mathbb{R}_q \cap [-a,a]$, we have

$$|\psi_{\alpha,q}(t,z)| \leq 2e^{(1+\sqrt{q})a|z|}.$$

So, for all $z \in \mathbb{C}$, $|f(z)| \leq \dfrac{2c_{\alpha,q}}{[2\alpha+2]_q} a^{2\alpha+2} \|u\|_{\infty,q} \;\; e^{(1+\sqrt{q})a|z|}$. ∎

REMARK. Since $\mathcal{D}_q([-a,a]) \subset \mathcal{S}_q(\mathbb{R}_q)$, then the Plancherel theorem and the previous proposition assert that $PW_{q,a}^{\alpha}$ is a non trivial subspace of $\mathcal{S}_q(\mathbb{R}_q)$.

The two following results give some characterizations of the q-Dunkl Paley-Wiener space $PW_{q,a}^{\alpha}$.

THEOREM 1. *Let $f \in \mathcal{S}_q(\mathbb{R}_q)$. The following three statements are equivalent:*

(1) $f \in PW^{\alpha}_{q,a}$;

(2) for all $n > 2\alpha + 2$ *there exists* $c_n > 0$ *such that*

$$|\Lambda^k_{\alpha,q} f(x)| \leq \frac{c_n}{1+|x|^n} a^k, \text{ for all } x \in \mathbb{R}_q \text{ and all } k \in \mathbb{N}; \qquad (19)$$

(3) there exist $n > 2\alpha + 2$ *and* $c_n > 0$ *satisfying (19).*

Proof. That (2) implies (3) is evident. It remains to prove that (1) implies (2) and (3) implies (1).

To prove that (1) implies (2), let $f \in PW^{\alpha}_{q,a}$ and $u \in \mathcal{D}_q([-a,a])$ satisfying (18).

For all $k, n \in \mathbb{N}$, we have $\Lambda^k_{\alpha,q} f(x) = \frac{c_{\alpha,q}}{2}(i)^k \int_{-a}^{a} u(t) t^k \psi_{\alpha,q}(t,x) |t|^{2\alpha+1} d_q t$, and by using (9) and (1), we get for all $x \in \mathbb{R}_q$,

$$x^n \Lambda^k_{\alpha,q} f(x) \quad = \quad \frac{c_{\alpha,q}}{2}(i)^{k-n} \int_{-\infty}^{\infty} \Lambda^n_{\alpha,q}\left[u(t)t^k\right] \psi_{\alpha,q}(t,x)|t|^{2\alpha+1} d_q t.$$

Since $u \in \mathcal{D}_q([-a,a])$, we have for all $k \in \mathbb{N}$, the function $u(t)t^k$ belongs to $\mathcal{D}_q(\mathbb{R}_q)$. Then, the fact that $\mathcal{D}_q(\mathbb{R}_q)$ is invariant by $\Lambda_{\alpha,q}$ implies that $\Lambda^n_{\alpha,q}\left[u(t)t^k\right]$ belongs to $\mathcal{D}_q(\mathbb{R}_q)$, for all $k, n \in \mathbb{N}$.

So, for $k < n$, we get by the help of (7), for all $x \in \mathbb{R}_q$,

$$|x|^n |\Lambda^k_{\alpha,q} f(x)| \leq \frac{2c_{\alpha,q}}{(q;q)_\infty} \int_{-\infty}^{\infty} \left|\Lambda^n_{\alpha,q}\left[u(t)t^k\right]\right| |t|^{2\alpha+1} d_q t = \widetilde{c}_{n,k} = (\widetilde{c}_{n,k}\, a^{-k})\, a^k.$$

From the definition of the operator $\Lambda_{\alpha,q}$, one can prove, by induction, that for all $n \in \mathbb{N}$, there exists a sequence $(s_m(\epsilon,n,q))_{-n \leq m \leq n, \epsilon = \pm 1}$ of real numbers such that for all function g,

$$\Lambda^n_{\alpha,q}\left[g(t)\right] = \frac{1}{t^n} \sum_{m=-n,\epsilon=\pm1}^{n} s_m(\epsilon,n,q) g\left(\epsilon q^m t\right).$$

So, for all $k, n \in \mathbb{N}$, we have

$$\Lambda^n_{\alpha,q}\left[u(t)t^k\right] = \frac{1}{t^n} \sum_{m=-n,\epsilon=\pm1}^{n} s_m(\epsilon,n,q) \left[u(\epsilon q^m t)(\epsilon q^m t)^k\right].$$

Since the function $t \mapsto \left[u(\epsilon q^m t)(\epsilon q^m t)^k\right], \quad -n \leq m \leq n$, has compact support in $\left[-q^{-|m|}a, q^{-|m|}a\right]$, then for $k \geq n$, we have

$$\left|\frac{1}{t^n} \sum_{m=-n,\epsilon=\pm1}^{n} s_m(\epsilon,n,q) \left[u(\epsilon q^m t)(\epsilon q^m t)^k\right]\right|$$

$$\leq \|u\|_{\infty,q} \sum_{m=-n,\epsilon=\pm 1}^{n} |s_m(\epsilon,n,q)| q^{mk}(q^{-m}a)^{k-n} = C_n a^{k-n}.$$

Hence, for $k \geq n$,

$$\left|x^n \Lambda_{\alpha,q}^k f(x)\right| = \left|\frac{c_{\alpha,q}}{2}(-i)^{k-n}\int_{-\infty}^{\infty} \Lambda_{\alpha,q}^n \left[u(t)t^k\right] \psi_{\alpha,q}(t,-x)|t|^{2\alpha+1} d_q t\right| \leq$$

$$\left(\frac{2c_{\alpha,q}}{(q;q)_\infty} C_n a^{-n} \int_{-q^{-n}a}^{q^{-n}a} |t|^{2\alpha+1} d_q t\right) a^k = \left(\frac{4c_{\alpha,q}}{(q;q)_\infty [2\alpha+2]_q} C_n a^{-n} (q^{-n}a)^{2\alpha+2}\right) a^k.$$

Finally, by taking $\widetilde{c}_n = \max\left\{\sup_{0\leq i\leq n} \widetilde{c}_{n,i}, \frac{4c_{\alpha,q}}{(q;q)_\infty[2\alpha+2]_q} C_n a^{-n}(q^{-n}a)^{2\alpha+2}\right\}$,

we get for all $n,k \in \mathbb{N}$ and all $x \in \mathbb{R}_q$, $\left|x^n \Lambda_{\alpha,q}^k f(x)\right| \leq \widetilde{c}_n a^k$.

Thus, for all $n,k \in \mathbb{N}$ and all $x \in \mathbb{R}_q$,

$$(1+|x|^n)\left|\Lambda_{\alpha,q}^k f(x)\right| \leq c_n a^k, \text{ with } c_n = \widetilde{c}_0 + \widetilde{c}_n.$$

Now, suppose that f satisfies the third statement of the theorem, put $u = F_D^{\alpha,q}(f)$ and fix $x \in \mathbb{R}_q$, such that $|x| > a$.

By the use of the relations (7) and (9), we obtain for all $k \in \mathbb{N}$,

$$F_D^{\alpha,q}\left(\Lambda_{\alpha,q}^k f\right)(x) = (i)^k x^k F_D^{\alpha,q}(f)(x) = (ix)^k u(x)$$

and

$$\begin{aligned}\left|F_D^{\alpha,q}\left(\Lambda_{\alpha,q}^k f\right)(x)\right| &\leq \frac{c_{\alpha,q}}{2}\int_{-\infty}^{\infty} |\Lambda_{\alpha,q}^k f(t)||\psi_{\alpha,q}(t,-x)||t|^{2\alpha+1} d_q t \\ &\leq \frac{2c_{\alpha,q}}{(q;q)_\infty} a^k \int_{-\infty}^{\infty} \frac{|t|^{2\alpha+1}}{1+|t|^n} d_q t.\end{aligned}$$

Then for all $k \in \mathbb{N}$,

$$|u(x)| \leq \left[\frac{2c_{\alpha,q}}{(q;q)_\infty}\int_{-\infty}^{\infty}\frac{|t|^{2\alpha+1}}{1+|t|^n} d_q t\right]\left(\frac{a}{|x|}\right)^k.$$

As $|x| > a$, we obtain by letting k to $+\infty$, $u(x) = 0$. This proves that $u \in \mathcal{D}_q([-a,a])$ and $f = (F_D^{\alpha,q})^{-1}(u) \in PW_{q,a}^\alpha$. ∎

Theorem 2. *The q-Dunkl Paley-Wiener space $PW_{q,a}^\alpha$ is the subspace of functions $f \in \mathcal{S}_q(\mathbb{R}_q)$ such that for all $x \in \mathbb{R}_q$, the function $T_z^{\alpha,q} f(x)$ is entire on $\mathbb{C}$ and for some $n > 2\alpha+1$ there exists $c_n > 0$, such that*

$$|T_z^{\alpha,q} f(x)| \leq \frac{c_n}{1+|x|^n}\psi_{\alpha,q}(|z|,-ia), \text{ for all } x \in \mathbb{R}_q \text{ and all } z \in \mathbb{C}.$$

Proof. Let $f \in PW_{q,a}^\alpha$. Then there exists $u \in \mathcal{D}_q([-a,a])$ such that $f = (F_D^{\alpha,q})^{-1}(u)$. So, by the relations (9) and (11), we have for all $k \in \mathbb{N}$ and all $x \in \mathbb{R}_q$,

$$\left|\Lambda_{\alpha,q}^k(f)(x)\right| = \frac{c_{\alpha,q}}{2}\left|\int_{-a}^{a} t^k u(t)\psi_{\alpha,q}(t,x)|t|^{2\alpha+1} d_q t\right| \leq C a^k.$$

Hence, Corollary 1 implies that for all $x \in \mathbb{R}_q$ the function $T_z^{\alpha,q} f(x)$ is entire on $\mathbb{C}$ and for all $z \in \mathbb{C}$,

$$|T_z^{\alpha,q} f(x)| \leq \sum_{k=0}^{+\infty} a_k(\alpha, q^2) \left|\Lambda_{\alpha,q}^k(f)(x)\right| |z|^k.$$

This inequality together with Theorem 1 implies that there exist $n > 2\alpha+2$ and $c_n > 0$, such that for all $x \in \mathbb{R}_q$ and all $z \in \mathbb{C}$,

$$|T_z^{\alpha,q} f(x)| \leq \frac{c_n}{1+|x|^n} \sum_{k=0}^{+\infty} a_k(\alpha, q^2)|az|^k = \frac{c_n}{1+|x|^n} \psi_{\alpha,q}(|z|, -ia).$$

Conversely, suppose that $f \in \mathcal{S}_q(\mathbb{R}_q)$, is satisfying: for all $y \in \mathbb{R}_q$, $T_z^{\alpha,q} f(y)$ is entire on $\mathbb{C}$ and there exist $n > 2\alpha + 2$ and $c_n > 0$, such that

$$|T_z^{\alpha,q} f(y)| \leq \frac{c_n}{1+|y|^n} \psi_{\alpha,q}(|z|, -ia), \quad \forall y \in \mathbb{R}_q, \forall z \in \mathbb{C}.$$

Let $b \in \mathbb{R}_q$ such that $|b| > a$. From the relation (6), we can see that the two functions $\int_0^{|b|} \psi_{\alpha,q}(t, y)|t|^{2\alpha+1} d_q t$ and $\int_0^{|b|} \psi_{\alpha,q}(t, -y)|t|^{2\alpha+1} d_q t$ are in $\mathcal{S}_q(\mathbb{R}_q)$, and from the relation (16), we can show that for all $x \in \mathbb{R}_q$,

$$T_x^{\alpha,q}\left[\int_0^{|b|} \psi_{\alpha,q}(t, y)|t|^{2\alpha+1} d_q t\right] = \int_0^{|b|} \psi_{\alpha,q}(t, x)\psi_{\alpha,q}(t, y)|t|^{2\alpha+1} d_q t. \quad (20)$$

Now, on the one hand, the conditions satisfying by f imply that for all $R > 0$ and all $z \in \mathbb{C}$ such that $|z| \leq R$, we have:

$$\left|T_z^{\alpha,q} f(y)\left[\int_0^{|b|} \psi_{\alpha,q}(t, y)|t|^{2\alpha+1} d_q t\right]\right| \leq \frac{4|b|^{2\alpha+2}}{(q;q)_\infty [2\alpha+2]_q} \frac{c_n}{1+|y|^n} \psi_{\alpha,q}(R, -ia),$$

the functions

$$\varphi_\pm(z) = \frac{c_{\alpha,q}}{2} \int_{-\infty}^{\infty} T_z^{\alpha,q} f(y) \left[\int_0^{|b|} \psi_{\alpha,q}(t, \pm y)|t|^{2\alpha+1} d_q t\right] |y|^{2\alpha+1} d_q y$$

are entire on $\mathbb{C}$ and for all $z \in \mathbb{C}$,

$$|\varphi_\pm(z)| \leq C_n \psi_{\alpha,q}(|z|, -ia), \quad (21)$$

with $C_n = \dfrac{2|b|^{2\alpha+2} c_{\alpha,q}}{(q;q)_\infty [2\alpha+2]_q} \displaystyle\int_{-\infty}^{\infty} \frac{c_n}{1+|y|^n} |y|^{2\alpha+1} d_q y$.

On the other hand, from the relations (17) and (20), the Fubini theorem and the fact that for all $x \in \widetilde{\mathbb{R}}_q$, all $y \in \mathbb{R}_q$ and all $t \in \mathbb{R}_q$ such that $0 \leq t \leq |b|$, we have

$$\left|f(y)\psi_{\alpha,q}(t, \pm y)\psi_{\alpha,q}(t, \pm x)|yt|^{2\alpha+1}\right| \leq \frac{16|b|^{2\alpha+1} c_n}{(q;q)_\infty^2} \frac{|y|^{2\alpha+1}}{1+|y|^n} \text{ and } n > 2\alpha+2,$$

one can see that for all $x \in \widetilde{\mathbb{R}}_q$,

$$\varphi_\pm(x) \;=\; \int_0^{|b|} F_D^{\alpha,q}(f)(\mp t)\psi_{\alpha,q}(t,\pm x)|t|^{2\alpha+1}d_qt.$$

Furthermore, it is not hard to prove that $\int_0^{|b|} F_D^{\alpha,q}(f)(\mp t)\psi_{\alpha,q}(t,\pm z)|t|^{2\alpha+1}d_qt$ are entire on $\mathbb{C}$, since for all $0 < t \le |b|$, $\psi_{\alpha,q}(t,.)$ is entire on $\mathbb{C}$, and for all $R > 0$ and all $z \in \mathbb{C}$ such that $|z| < R$,

$$\left|F_D^{\alpha,q}(f)(\mp t)\psi_{\alpha,q}(t,\pm z)|t|^{2\alpha+1}\right| \le |b|^{2\alpha+1}\|F_D^{\alpha,q}(f)\|_{\infty,q}\psi_{\alpha,q}(R,-i|b|).$$

But 0 is a limit point of $\widetilde{\mathbb{R}}_q$, so the analytic theorem shows that for all $z \in \mathbb{C}$,

$$\varphi_\pm(z) = \int_0^{|b|} F_D^{\alpha,q}(f)(\mp t)\psi_{\alpha,q}(t,\pm z)|t|^{2\alpha+1}d_qt.$$

Hence, from (21),we obtain, since $a, b \in \mathbb{R}_q$ and $|b| > a$, for all $z \in \mathbb{C}$

$$\left|\int_0^{|b|} F_D^{\alpha,q}(f)(\mp t)\psi_{\alpha,q}(t,\pm z)|t|^{2\alpha+1}d_qt\right| \le C_n\psi_{\alpha,q}(|z|,-ia) \le C_n\psi_{\alpha,q}(|z|,-iq|b|).$$

So, for all $z \in \mathbb{C}$,

$$|b|^{2\alpha+2}\left|F_D^{\alpha,q}(f)(\mp|b|)\psi_{\alpha,q}(|b|,\pm z)\right| - \left|\sum_{k=1}^{\infty} F_D^{\alpha,q}(f)(\mp|b|q^k)\psi_{\alpha,q}(|b|q^k,\pm z)\right.$$
$$\left.\times\, |bq^k|^{(2\alpha+2)}\right| \;\le\; \frac{C_n}{1-q}\psi_{\alpha,q}(|z|,-iq|b|).$$

Moreover, $F_D^{\alpha,q}(f)$ is bounded on $\mathbb{R}_q$, then using the fact that for all $z \in \mathbb{C}$ and all positive integer k, $\left|\psi_{\alpha,q}(|b|q^k,\pm z)\right| \le \psi_{\alpha,q}(|z|,-iq|b|)$, we get

$$\left|F_D^{\alpha,q}(f)(b)\right| \le \widetilde{C}\;\; \frac{\psi_{\alpha,q}(|z|,-iq|b|)}{|\psi_{\alpha,q}(|b|,\pm z)|}.$$

A replacement of z by ix or $-ix$ gives

$$\left|F_D^{\alpha,q}(f)(b)\right| \le \widetilde{C}\;\; \frac{\psi_{\alpha,q}(x,-iq|b|)}{\psi_{\alpha,q}(x,-i|b|)},\ \text{for all } x \in \mathbb{R}_{q,+}.$$

But,

$$x^{-2}\left[\psi_{\alpha,q}(x,-i|b|) - 1 - |b|x\right] = \sum_{k=2}^{\infty} a_k(\alpha,q^2)(x|b|)^{(k-2)}$$
$$= \sum_{k=1}^{\infty} a_{2k}(\alpha,q^2)(x|b|)^{(2k-2)} + \sum_{k=1}^{\infty} a_{2k+1}(\alpha,q^2)(x|b|)^{(2k-1)}$$
$$\ge \psi_{\alpha,q}(x,-iq|b|)$$

and $\lim_{x\to\infty} \psi_{\alpha,q}(x, -i|b|) = \infty$, then

$$\lim_{x\to\infty} \frac{\psi_{\alpha,q}(x,-i|b|) - 1 - |b|x}{\psi_{\alpha,q}(x,-i|b|)} = 1 \quad \text{and} \quad \lim_{x\to\infty} \frac{\psi_{\alpha,q}(x,-iq|b|)}{\psi_{\alpha,q}(x,-i|b|)} = 0.$$

Thus $u(b) = F_D^{\alpha,q}(f)(\pm|b|) = 0$, which proves that $u = F_D^{\alpha,q}(f) \in \mathcal{D}_q([-a,a])$ and as consequence $f \in PW_{q,a}^{\alpha}$. ∎

In the end of this section, using the relation (8) and making a proof as in [5, Theorem 4], we establish the following result, which gives a q-sampling formula for the q-Dunkl bandlimited signal, using $\pm q^n$, $n \in \mathbb{Z}$, as sampling points.

THEOREM 3. *For $f \in PW_{q,a}^{\alpha}$, we have*

$$f(z) = (1-q)\frac{2a^{2\alpha+2}}{[2\alpha+2]_q}\frac{(c_{\alpha,q})^2}{2}\sum_{k\in\mathbb{Z},\varepsilon=\pm1} f(\varepsilon q^k)E(z,a,\varepsilon,k,q),$$

where: $E(z,a,\varepsilon,k,q)$

$$= q^{k(2\alpha+2)}\left[\frac{\varepsilon q^k j_{\alpha+1}(\varepsilon a q^k, q^2) j_\alpha(aq^{-1}z, q^2) - z j_{\alpha+1}(az, q^2) j_\alpha(\varepsilon a q^{-1} q^k, q^2)}{\varepsilon q^k + z}\right].$$

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